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ON THE LOCAL GEOMETRY OF GRAPHS IN TERMS OF THEIR SPECTRA

BRICE HUANG

*Department of Mathematics, Massachusetts Institute of Technology,
Cambridge, MA, USA.*

MUSTAZEE RAHMAN

*Department of Mathematics, Massachusetts Institute of Technology,
Cambridge, MA, USA and
Department of Mathematics, KTH Royal Institute of Technology,
Stockholm, Sweden.*

ABSTRACT. In this paper we consider the relation between the spectrum and the number of short cycles in large graphs. Suppose G_1, G_2, G_3, \dots is a sequence of finite and connected graphs that share a common universal cover T and such that the proportion of eigenvalues of G_n that lie within the support of the spectrum of T tends to 1 in the large n limit. This is a weak notion of being Ramanujan. We prove such a sequence of graphs is asymptotically locally tree-like. This is deduced by way of an analogous theorem proved for certain infinite sofic graphs and unimodular networks, which extends results for regular graphs and certain infinite Cayley graphs.

E-mail addresses: `bmhuang@mit.edu`, `mustazee@gmail.com`.

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1. INTRODUCTION

This paper is about how the spectrum of a big, bounded degree graph determines its local geometry around typical vertices. For a finite and connected graph G , let

$$\lambda_1(G) > \lambda_2(G) \geq \lambda_3(G) \geq \dots$$

be the eigenvalues of its adjacency matrix. Let T be the universal cover tree of G and denote by $\rho(T)$ its spectral radius, which is the operator norm of the adjacency matrix of T acting on $\ell^2(T)$. If G is d -regular then $\lambda_1(G) = d$ and $\rho(T) = 2\sqrt{d-1}$, T being the d -regular tree.

It is easy to see that $\lambda_1(G) \geq \rho(T)$. Various extensions of the Alon-Boppana Theorem state that for every $\varepsilon > 0$, a positive proportion of the eigenvalues of G lie outside the interval $[-\rho(T) + \varepsilon, \rho(T) - \varepsilon]$ independently of the size of G ; see e.g. [6, 8, 10, 19, 20]. What happens when the eigenvalues actually concentrate within $[-\rho(T), \rho(T)]$?

A graph G is Ramanujan if $|\lambda_i(G)| \leq \rho(T)$ for every $i \geq 2$. It is a fairly well-understood theme that large, d -regular Ramanujan graphs locally resemble the d -regular tree in that they contain few short cycles. For an illustration of such results see [1, 5, 12, 15, 16] and references therein. This relation is not as well understood for sparse, irregular graphs. We prove the following with regards to the spectrum and local geometry.

Suppose G_1, G_2, G_3, \dots is a sequence of finite and connected graphs. They are *weakly Ramanujan* if they have a common universal cover tree T and if, counting with multiplicity,

$$(1.1) \quad \frac{\#\{\text{Eigenvalues of } G_n \text{ s.t. } |\lambda_i(G_n)| \leq \rho(T)\}}{|G_n|} \longrightarrow 1 \text{ as } n \rightarrow \infty.$$

Theorem 1. *Consider a sequence of weakly Ramanujan graphs as in (1.1). Suppose that $|G_n| \rightarrow \infty$. Then the graphs are asymptotically locally tree-like in that for every $r > 0$,*

$$\frac{\#\{\text{Vertices } v \in G_n \text{ s.t. its } r\text{-neighbourhood is a tree}\}}{|G_n|} \longrightarrow 1 \text{ as } n \rightarrow \infty.$$

Let us make a few remarks about Theorem 1. First, if the r -neighbourhood of a vertex $v \in G$ is a tree then it agrees with the r -neighbourhood of any vertex \hat{v} in the universal cover of G that maps to v under the cover map. So roughly speaking, large weakly Ramanujan graphs locally look like their universal covers around most vertices.

Second, it is natural in our context to assume a sequence of graphs share a common universal cover. For one thing it is a generalization of a sequence of d -regular graphs, (a, b) -biregular graphs, etc. But more so, it provides a way to compare the spectrum and geometry of graphs with differing sizes on a common scale. For example, two finite graphs with the same universal cover have the same degree distribution, hence, also the same average and maximal degree. They also have the same maximal eigenvalue according to a theorem in

[14]. The definition of Ramanujan graphs in terms of their universal covers was introduced in [8] (stated also in [12]).

The version of Theorem 1 for d -regular weakly Ramanujan graphs has been proved in [1]. In this case many tools, such as the Green's function and spectral measure of the d -regular tree, are available in precise form. This is lacking for general universal covers where even the computation of the spectral radius is difficult (although an algorithm is provided in [18] and various bounds are given in [11, 19]).

Theorem 1 is also motivated by the classical work of Kesten which relates the geometry of groups to their spectra [13]. Namely, the d -regular Cayley graph of a finitely generated group is amenable if and only if its spectral radius is the degree d . In the other direction, what happens to the geometry when the spectrum is “as small as possible”, which is to say that the spectral radius is $2\sqrt{d-1}$? One can ask this question for finite graphs, bounded degree sofic graphs, or more generally, for unimodular networks. Theorem 1 considers a finitary case and Theorem 3 below certain sofic graphs. The proof of Theorem 3 also applies to analogous unimodular networks.

The assumption in Theorem 1 that $|G_n| \rightarrow \infty$ is necessary. For example, if all the graphs are equal to a common cyclic graph then the sequence is weakly Ramanujan. (The universal cover is \mathbb{Z} with $\rho(\mathbb{Z}) = 2$ and all the eigenvalues lie in the interval $[-2, 2]$.) However, this is the only obstruction as being weakly Ramanujan implies $|G_n| \rightarrow \infty$ so long as the common average degree of the graphs is larger than 2. This follows from the following theorem, which asserts that $\lambda_1(G_n) > \rho(T)$ when the common average degree is more than 2.

Theorem 2. *Let G be a finite and connected graph with universal cover T . Then $\lambda_1(G) = \rho(T)$ if and only if G has at most one cycle or, equivalently, if and only if the average degree of G is at most 2.*

This theorem is somewhat of an analogue of Theorem 1 for a single finite graph. The proof turns out to be a bit delicate. For instance, consider the bowtie graph G obtained by gluing together two triangles at a common vertex. It has $\lambda_1(G) = (1 + \sqrt{17})/2$ and $\rho(T) = (\sqrt{3} + \sqrt{11})/2$. The spectral gap is about 0.04 and the average degree is also smaller than $\rho(T)$. In general, the spectral gap can be arbitrarily small for graphs formed by gluing together two large cycles at a common vertex. The proof of Theorem 2 is based on orienting and weighting the edges of a finite graph in a way that allow to compare the norm of its adjacency matrix with the cover's via the Rayleigh variational principle. We have learned it is related to “Gabber lemma”, see [9].

Theorem 1 is proved in the following section. The idea behind the proof is to reduce the theorem to an analogous theorem about certain infinite random rooted graphs (sofic graphs and unimodular networks) by using the notion of local convergence of graphs. The key result of the paper is a proof of the analogue of Theorem 1 for these infinite graphs,

which is Theorem 3 below, proved in Section 3. The proof establishes a lower bound on the spectral radius of such graphs in terms of a probabilistic notion of cycle density. Theorem 2 is then proved in Section 4. Section 5 concludes with some questions.

2. A REDUCTION OF THEOREM 1

We begin with the notion of local convergence of graphs and some of its properties used for the proof of Theorem 1. A complete account, including proofs, may be found in [2, 3, 4, 7].

Let $B_r(G, v)$ be the r -neighbourhood of a vertex v in a graph G . A sequence of finite and connected graphs G_1, G_2, G_3, \dots converges locally if the following holds. For every r and every rooted, connected graph (H, o) having radius at most r from the root o , the ratio

$$\frac{\#\{\text{Vertices } v \in G_n \text{ s.t. } B_r(G_n, v) \cong (H, o)\}}{|G_n|} \text{ converges as } n \rightarrow \infty.$$

The isomorphism relation \cong is for rooted graphs, i.e. the isomorphism must take the root of one to the other. Local convergence is also known as Benjamini-Schramm convergence as it was formulated by them in [3].

A locally convergent sequence of graphs may be represented as a random rooted graph in the following way. Let \mathcal{G} be the set of all rooted and connected graphs whose vertex sets are subsets of the integers. Identify the graphs in \mathcal{G} up to their rooted isomorphism class. The set \mathcal{G} is a complete and separable metric space with the distance between (H, o) and (H', o') being 2^{-r} , where r is the maximal integer such that $B_r(H, o) \cong B_r(H', o')$. A random rooted graph is simply a Borel probability measure on \mathcal{G} or, in other words, a \mathcal{G} -valued random variable (\mathbf{G}, o) that is Borel-measurable. Given a locally convergent sequence of graphs as above, there is a random rooted graph (\mathbf{G}, o) such that for every r and (H, o) as above,

$$\frac{\#\{\text{Vertices } v \in G_n \text{ s.t. } B_r(G_n, v) \cong (H, o)\}}{|G_n|} \longrightarrow \mathbf{Pr}[B_r(\mathbf{G}, o) \cong (H, o)].$$

A random rooted graph that is obtained from a locally convergent sequence of finite graphs is called *sofic*. Examples include Cayley graphs of amenable groups such as \mathbb{Z}^d , regular trees, as well as Cayley graphs of residually finite groups. More examples may be found in [2, 3, 4, 7]. Sofic graphs satisfy an important property known as the *mass transport principle*, as we explain.

Suppose (\mathbf{G}, o) is sofic. Consider a bounded and measurable function $F(G, u, v)$ defined over doubly rooted graphs such that it depends only on the double-rooted isomorphism class of (G, u, v) . The mass transport principle states that

$$\mathbf{E} \sum_{v \in \mathbf{G}} F(\mathbf{G}, o, v) = \mathbf{E} \sum_{v \in \mathbf{G}} F(\mathbf{G}, v, o).$$

The above is readily verified for a finite graph rooted at a uniformly random vertex, and it continues to hold in the local limit, which is why it holds for a sofic graph. Random rooted graphs that satisfy the mass transport principle are called unimodular networks. It is not known whether every unimodular network is a sofic graph.

Let us describe the universal cover of a sofic graph. Recall the universal cover of a graph G is the unique tree T for which there is a surjective graph homomorphism $\pi : T \rightarrow G$, called the cover map, such that π is locally bijective: for every $\hat{v} \in T$, π provides a bijection $B_1(T, \hat{v}) \xrightarrow{\pi} B_1(G, \pi(\hat{v}))$. If $\pi(\hat{v}) = \pi(\hat{u})$ then the rooted graphs $(T, \hat{v}) \cong (T, \hat{u})$. Therefore, the universal cover of a sofic graph (\mathbf{G}, o) may be defined as its samplewise universal cover (\mathbf{T}, \hat{o}) , where \hat{o} is any vertex that is mapped to o by the cover map.

The spectral radius of a sofic graph (\mathbf{G}, o) is defined as follows. Let $W_k(G, o)$ be the set of closed walks of length k from o in a graph G and denote by $|W|_k(G, o)$ its size. The spectral radius of (\mathbf{G}, o) is

$$\rho(\mathbf{G}) = \lim_{k \rightarrow \infty} (\mathbf{E} |W|_{2k}(\mathbf{G}, o))^{1/2k}.$$

Recall that the spectral radius of a connected graph G is also the exponential growth rate of $|W|_{2k}(G, v)$ for any vertex v . The relation of $\rho(\mathbf{G})$ to the adjacency matrix of \mathbf{G} is that it equals the sup norm, $\|\rho(\mathbf{G}, o)\|_\infty$, of the samplewise spectral radius of (\mathbf{G}, o) . It is also the largest element in the support of the “averaged” spectral measure of (\mathbf{G}, o) , which is a probability measure over the reals with moments $\mathbf{E} |W|_0(\mathbf{G}, o), \mathbf{E} |W|_1(\mathbf{G}, o), \mathbf{E} |W|_2(\mathbf{G}, o), \dots$

We prove the following regarding the spectrum and geometry of an infinite sofic graph.

Theorem 3. *Let (\mathbf{G}, o) be an almost surely infinite sofic graph that is the local limit of a sequence of finite graphs sharing a common universal cover T . If $\rho(\mathbf{G}) = \rho(T)$ then \mathbf{G} is isomorphic to T almost surely.*

This theorem (and its proof) may be extended to unimodular networks whose universal cover is deterministic and quasi-transitive.

2.1. Reducing Theorem 1 to sofic graphs. The theorem may be reformulated as asserting that given a sequence of weakly Ramanujan graphs sharing a common universal cover T , and with their sizes tending to infinity, the sequence converges locally to T . The root of T will be a random vertex whose distribution is determined by the convergent sequence, although it does not depend on the particular sequence.

More precisely, as T is the universal cover of a finite graph, it is quasi-transitive. Let

$$(2.1) \quad \{\hat{v}_1, \hat{v}_2, \dots, \hat{v}_m\}$$

be vertices that make a set of representatives for $T/\text{Aut}(T)$. There is a probability measure (p_1, \dots, p_m) on them such that if vertex \hat{o} is chosen according to it then (T, \hat{o}) satisfies the

mass transport principle; see [2]. Any sequence of graphs converging locally to T has limit (T, \hat{o}) . Moreover, if a finite graph G is covered by T then p_j is the proportion of vertices $v \in G$ such that the vertices of $\pi^{-1}(v)$ can be sent to \hat{v}_j by automorphisms of T (all vertices in $\pi^{-1}(v)$ belong to the same orbit); see [17].

Suppose G_1, G_2, G_3, \dots is a sequence of weakly Ramanujan graphs as in the statement of the theorem. Since they share a common universal cover, their vertex degrees are bounded by some integer Δ . By a simple diagonalization argument (there are at most Δ^{r+1} rooted graphs of radius $\leq r$ and maximal degree $\leq \Delta$), the sequence is pre-compact in the local topology. We must prove that its only limit point is (T, \hat{o}) .

Suppose (\mathbf{G}, o) is a limit point of the sequence. Then its universal cover is isomorphic to (T, \hat{o}) almost surely. This is because the sequence has a common universal cover and the universal cover is a continuous mapping of its base graph in the local topology; see [17]. Theorem 1 is proved if \mathbf{G} is isomorphic to T almost surely as unrooted graphs. The mass transport principle specifies the distribution of the root as we have explained. By Lemma 2.1 below, $\rho(\mathbf{G}) = \rho(T)$. The theorem now follows from Theorem 3.

Lemma 2.1. *Let G_1, G_2, G_3, \dots be a locally convergent sequence of weakly Ramanujan graphs. Suppose (\mathbf{G}, o) is its limit and T is the common universal cover. Then $\rho(\mathbf{G}) = \rho(T)$.*

Proof. Let Δ be the maximal vertex degree of the graph sequence. Due to local convergence and bounded convergence theorem, $\mathbf{E} |W|_{2k}(\mathbf{G}, o)$ is the limit of $\frac{1}{|G_n|} \sum_{v \in G_n} |W|_{2k}(G_n, v)$. Let q_n be the proportion of eigenvalues of G_n that are at most $\rho(T)$ in absolute value, so then $q_n \rightarrow 1$. Note that all eigenvalues of G_n are bounded by Δ in absolute value. The aforementioned average is the trace of the $(2k)$ -th power of the adjacency matrix of G_n , normalized by $|G_n|$. Thus,

$$\frac{1}{|G_n|} \sum_{v \in G_n} |W|_{2k}(G_n, v) \leq q_n \rho(T)^{2k} + (1 - q_n) \Delta^{2k}.$$

Upon taking limits we conclude that $\rho(\mathbf{G}) \leq \rho(T)$.

For the inequality in the other direction, recall the vertices $\hat{v}_1, \dots, \hat{v}_m$ from (2.1) and the associated probability measure (p_1, \dots, p_m) on them. Suppose a finite graph G has universal cover T . Recall p_j is the proportion of vertices in G that have a pre-image in T , under the cover map, which can be sent to \hat{v}_j by a T -automorphism. If $\hat{v} \in T$ is mapped to $v \in G$ by the cover map then $|W|_{2k}(G, v) \geq |W|_{2k}(T, \hat{v})$. Indeed, the cover map provides an injection from $W_{2k}(T, \hat{v})$ into $W_{2k}(G, v)$ due to its path lifting property. Consequently,

$$\frac{1}{|G|} \sum_{v \in G} |W|_{2k}(G, v) \geq \sum_{j=1}^m p_j |W|_{2k}(T, \hat{v}_j).$$

Applying the inequality above to every G_n and taking the large n limit gives

$$\mathbf{E} |W|_{2k}(\mathbf{G}, o) \geq \sum_{j=1}^m p_j |W|_{2k}(T, \hat{v}_j).$$

Since $\rho(T)$ is the large k limit of $|W|_{2k}(T, \hat{v}_j)^{1/2k}$ for every \hat{v}_j , the inequality above implies that $\rho(\mathbf{G}) \geq \rho(T)$. \blacksquare

3. A SPECTRAL RIGIDITY THEOREM

Theorem 3 will be proved by showing that if there is an ℓ such that

$$\Pr[o \text{ lies in an } \ell - \text{cycle of } \mathbf{G}] > 0,$$

then $\rho(\mathbf{G})/\rho(T) \geq 1 + \delta$ for some positive δ . This result is built up in the subsequent sections by drawing a connection between the spectral radius of \mathbf{G} and of T in terms of the norms of certain Markov operators associated to random walks on the fundamental group of \mathbf{G} . This connection was established in [1].

3.1. Counting walks using the fundamental group. Consider a connected graph H which may be countably infinite and may have multi-edges and loops around its vertices. (A loop contributes degree 2 to its vertex.) Let $\pi(H, v)$ be its fundamental group based at vertex v , which consists of homotopy classes of closed walks from v under the operation of concatenation. It is a free group. (See [17] for an account on the fundamental group of graphs and its properties mentioned here.)

Let $W_k(u, v)$ be the set of walks in H of length k from u to v . Note $W_k(v, u) = W_k^{-1}(u, v)$, where the inverse means walking in the opposite direction. The set

$$WW^{-1} = \{PQ^{-1} : P, Q \in W_k(u, v)\}$$

consists of closed walks from u of length $2k$ and is closed under inversion. It naturally maps into $\pi(H, u)$, and the uniform measure on it pushes forward to a measure on the image $\overline{WW^{-1}} \subset \pi(H, u)$. Note the push-forward may not be uniform measure on the image as different closed walks in WW^{-1} may be homotopy equivalent.

Consider the random walk on $\pi(H, u)$ whose step distribution is the aforementioned push-forward measure of WW^{-1} . Since WW^{-1} is closed under taking inverses, this is a symmetric random walk on the Cayley graph of the subgroup of $\pi(H, u)$ generated by WW^{-1} . Denote the norm of its associated Markov operator by

$$(3.1) \quad \|M_k\|(u, v).$$

Now fix a vertex $o \in H$, a path P from o to u and another path Q from o to v . The set $PW_k(u, v)Q^{-1}$ consists of closed walks from o . Consider the random walk on $\pi(H, o)$

whose step distribution is the push-forward of the uniform measure on this set by its the natural mapping into $\pi(H, o)$. Denote by $\sqrt{||M_k||}(u, v)$ the norm of the Markov operator of this random walk. This operator may not be symmetric since the set $PW_k(u, v)Q^{-1}$ is not closed under taking inverses. However,

$$\sqrt{||M_k||}(u, v)^2 = ||M_k||(\pi(u, v))$$

because the norm in question is the square root of the norm of the Markov operator for the random walk on $\pi(H, o)$ associated to the set

$$(PW_k(u, v)Q^{-1})(PW_k(u, v)Q^{-1})^{-1} = PWW^{-1}P^{-1}.$$

The Markov operator for $PWW^{-1}P^{-1}$ is isomorphic – as an operator on $\ell^2(\pi(H, o))$ – to the Markov operator for WW^{-1} on $\ell^2(\pi(H, u))$. The isomorphism comes from the natural isomorphism of groups $\pi(H, o) \leftrightarrow \pi(H, u)$. The norm of the Markov operator for WW^{-1} is $||M_k||(\pi(u, v))$.

3.2. The counting argument. Let H be a graph as above. A *purely backtracking* walk in H is a closed walk that is homotopic to the empty walk, that is, it reduces to the identity in the fundamental group of H . Purely backtracking walks in H from a base point o are in one to one correspondence with closed walks in the universal cover of H from a base point \hat{o} such that $\pi(\hat{o}) = o$ (π being the cover map). This is due to the path lifting property of the universal cover map.

Choose an arbitrary vertex $o \in H$. Let n and k be arbitrary integers with nk being even. Denote by W all closed walks from o of length nk . Denote by N all purely backtracking walks from o of length nk . The following inequality is from [1].

$$(3.2) \quad \log |W| - \log |N| \geq \frac{1}{|N|} \sum_{P \in N} \sum_{j=1}^n -\frac{1}{2} \log ||M_k||(\pi(P_{(j-1)k}, P_{jk})).$$

The proof is based on partitioning the set W in the following way. Two walks in W are equivalent if their locations coincide at the times $0, k, 2k, \dots, nk$. Let W_N denote the set of walks in W that are equivalent to some purely backtracking walk. Observe that

$$|W_N| = \sum_{P \in N} \frac{|[P]|}{|[P] \cap N|}.$$

The term $|[P]|/|[P] \cap N|$ is the reciprocal of the probability that a uniform random walk in $[P]$ is purely backtracking. The probability can be interpreted in the following way. Consider the random walk on $\pi(H, o)$ whose step distribution is the push forward of the uniform measure on $[P] \mapsto \pi(H, o)$. The probability under consideration is the one-step return probability of this random walk. It may be expressed as $\langle M_P \text{id}, \text{id} \rangle$, where M_P is

the Markov operator of this random walk. Therefore,

$$|W| \geq \sum_{P \in N} \langle M_P \text{id}, \text{id} \rangle^{-1}.$$

Every $Q \in [P]$ agrees with P at the times $0, k, \dots, nk$. This allows us to decompose Q into petals as in Figure 1.

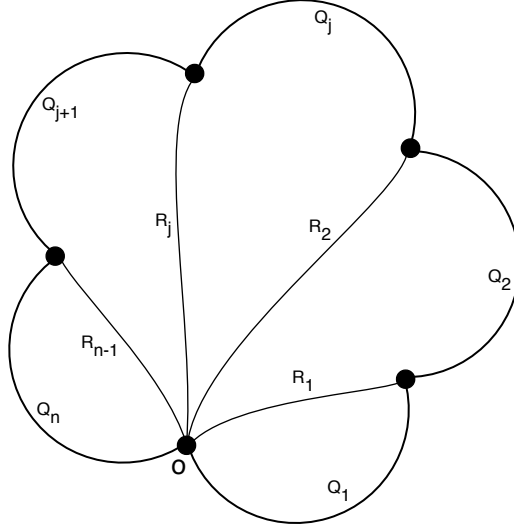


FIGURE 1. Decomposing a closed walk into petals.

Here, Q_j is the segment of Q from $Q_{(j-1)k} = P_{(j-1)k}$ to $Q_{jk} = P_{jk}$. R_j is a fixed path from o to P_{jk} chosen independently of Q . The decomposition is that

$$Q = \underbrace{(Q_1 R_1^{-1})}_{T_1} \cdot \underbrace{(R_1 Q_2 R_2^{-1})}_{T_2} \cdots \underbrace{(R_{n-1} Q_n)}_{T_n}.$$

Under this decomposition, a uniformly random element $Q \in [P]$ becomes the product $T_1 \cdots T_n$, where T_j is a uniformly random element of $R_{j-1} W_k(P_{(j-1)k}, P_{jk}) R_j^{-1}$. This uses that the locations of Q are pinned at the times $0, k, \dots, nk$.

Let M_j be the Markov operator for the random walk on $\pi(H, o)$ with step distribution T_j . Then $M_P = M_1 \cdots M_n$, and

$$\langle M_P \text{id}, \text{id} \rangle \leq \|M_P\| \leq \prod_j \|M_j\|.$$

Each $\|M_j\|$ equals $\sqrt{\|M_k\|}(P_{(j-1)k}, P_{jk})$. Therefore,

$$|W| \geq \sum_{P \in N} \prod_{j=1}^n \|M_k\|(P_{(j-1)k}, P_{jk})^{-1/2}.$$

Dividing the above by $|N|$, using the inequality of arithmetic-mean and geometric-mean, and then taking the logarithm gives the inequality from (3.2).

Let (\mathbf{G}, o) be an infinite sofic graph as in the statement of Theorem 3. The mass transport principle simplifies the right hand side of (3.2) for (\mathbf{G}, o) as follows.

Lemma 3.1. *In this setting the following equation holds for $j = 1, \dots, n$.*

$$\mathbf{E} \frac{1}{|N|_{nk}(\mathbf{G}, o)} \sum_{P \in N_{nk}(\mathbf{G}, o)} \log \|M_k\|(P_{(j-1)k}, P_{jk}) = \mathbf{E} \sum_{P \in N_{nk}(\mathbf{G}, o)} \frac{\log \|M_k\|(o, P_k)}{|N|_{nk}(\mathbf{G}, P_{(n-j+1)k})}.$$

Proof. Consider the function

$$F(H, u, v) = \frac{1}{|N|_{nk}(H, v)} \sum_{P \in N_{nk}(H, v)} \mathbf{1}_{\{P_{(j-1)k}=u\}} \log \|M_k\|(u, P_{jk}).$$

It depends on the doubly-rooted isomorphism class of (H, u, v) . Now,

$$\sum_{u \in H} F(H, u, v) = \frac{1}{|N|_{nk}(H, v)} \sum_{P \in N_{nk}(H, v)} \log \|M_k\|(P_{(j-1)k}, P_{jk}).$$

On the other hand,

$$F(H, u, v) = \frac{1}{|N|_{nk}(H, v)} \sum_{P \in N_{nk}(H, u)} \mathbf{1}_{\{P_{(n-j+1)k}=v\}} \log \|M_k\|(u, P_k)$$

because we can also sum over the walks by starting them at u instead of v . Therefore,

$$\sum_{v \in H} F(H, u, v) = \sum_{P \in N_{nk}(H, u)} \frac{\log \|M_k\|(u, P_k)}{|N|_{nk}(H, P_{(n-j+1)k})}.$$

The mass-transport principle for (\mathbf{G}, o) states that

$$\mathbf{E} \sum_{u \in \mathbf{G}} F(\mathbf{G}, u, o) = \mathbf{E} \sum_{v \in \mathbf{G}} F(\mathbf{G}, o, v),$$

which is the equation in the statement of the lemma. ■

3.3. Bounds. Applying the bound from (3.2) to (\mathbf{G}, o) , taking the expectation value, applying Lemma 3.1 and then dividing by nk gives

$$\frac{\mathbf{E} \log |W|_{nk}(\mathbf{G}, o) - \mathbf{E} \log |N|_{nk}(\mathbf{G}, o)}{nk} \geq \mathbf{E} \sum_{P \in N_{nk}(\mathbf{G}, o)} \frac{1}{n} \sum_{j=1}^n \frac{-(2k)^{-1} \log \|M_k\|(o, P_k)}{|N|_{nk}(\mathbf{G}, P_{(n-j+1)k})}.$$

The term $-(2k)^{-1} \log \|M_k\|(P_o, P_k)$ is non-negative. We would thus like to replace each of the terms $|N|_{nk}(\mathbf{G}, P_{(n-j+1)k})$ by $|N|_{nk}(\mathbf{G}, o)$, after which the average over the parameter j would be replaced by unity. Recall the universal cover of \mathbf{G} is the non-random tree T

and $W_{nk}(T, \hat{v}) = N_{nk}(\mathbf{G}, \pi(\hat{v}))$. Therefore, the cost of replacing $|N|_{nk}(\mathbf{G}, P_{(n-j+1)k})$ by $|N|_{nk}(\mathbf{G}, o)$ while preserving the \geq inequality is given by the multiplicative factor

$$r_{nk} = \min_{i,j} \frac{|W_{nk}|(T, \hat{v}_i)}{|W_{nk}|(T, \hat{v}_j)},$$

where $\hat{v}_1, \dots, \hat{v}_m$ are a set of orbit representatives for T as given by (2.1). Part 1 of the following lemma shows that $r_{nk} \geq \Delta^{-2d}$, where d is the maximum distance between any two of the \hat{v}_i 's and Δ is the maximal degree of T .

Lemma 3.2. *Let H be a connected graph having maximum degree at most Δ . Let x and y be two of its vertices having distance d between them.*

- (1) $|W|_{2k}(H, y) \leq \Delta^{2d} |W|_{2k}(H, x)$.
- (2) $|W|_{2k+2j}(H, x) \leq \Delta^{2j} |W|_{2k}(H, x)$.

Proof. Let A be the adjacency matrix of H acting on $\ell^2(H)$ (H may be countably infinite).

The inequality in (1) follows from the inequality in (2) upon observing that $|W|_{2k}(H, y) \leq |W|_{2k+2d}(H, x)$. For the proof of (2), we have $|W|_{2k+2j}(H, x) = \langle A^{2k+2j} \delta_x, \delta_x \rangle$ and the latter equals $\langle A^{2j} (A^k \delta_x), (A^k \delta_x) \rangle$. Thus,

$$|W|_{2k+2j}(H, x) \leq \|A^{2j}\| \langle A^k \delta_x, A^k \delta_x \rangle \leq \Delta^{2j} \langle A^k \delta_x, A^k \delta_x \rangle = \Delta^{2j} |W|_{2k}(H, x).$$

■

Lemma 3.3. *Let Δ be the maximal degree of T . The following inequality holds:*

$$\log \rho(\mathbf{G}) - \log \rho(T) \geq \sup_{k \geq 1} \frac{\mathbf{E} - \log \|M_{2k}\|(o, o)}{4k \Delta^{2d+2k}}.$$

Proof. Observe that \mathbf{G} has maximal degree Δ almost surely because it is covered by T . By Lemma 3.2,

$$(3.3) \quad \frac{\mathbf{E} \log |W|_{nk}(\mathbf{G}, o) - \mathbf{E} \log |N|_{nk}(\mathbf{G}, o)}{nk} \geq \Delta^{-2d} \mathbf{E} \frac{1}{|N|_{nk}(\mathbf{G}, o)} \sum_{P \in N_{nk}(\mathbf{G}, o)} \frac{-\log \|M_k\|(o, P_k)}{2k}.$$

The expectation on the right hand side of (3.3) is an average over (\mathbf{G}, o, P^n) , where P^n is a uniformly random purely backtracking walk in \mathbf{G} starting at o and having length nk . If k is even then $\mathbf{Pr}[P_k^n = o] \geq \Delta^{-k}$. This is because a purely backtracking walk from o of length nk will be at o at step k if it is a purely backtracking walk from o of length k followed by one of length $nk - k$. Consequently,

$$\mathbf{Pr}[P_k^n = o] \geq \mathbf{E} \frac{|N|_k(\mathbf{G}, o) \cdot |N|_{nk-k}(\mathbf{G}, o)}{|N|_{nk}(\mathbf{G}, o)} \geq \Delta^{-k},$$

where the last inequality is due to $|N|_k(\mathbf{G}, o) \geq 1$ and, also, by part 2 of Lemma 3.2, due to $|N|_{nk-k}(\mathbf{G}, o) \geq \Delta^{-k}|N|_{nk}(\mathbf{G}, o)$. Since $-\log \|M_k\|(u, v)$ is non-negative, (3.3) implies that for every even k ,

$$\frac{\mathbf{E} \log |W|_{nk}(\mathbf{G}, o) - \mathbf{E} \log |N|_{nk}(\mathbf{G}, o)}{nk} \geq \frac{\mathbf{E} - \log \|M_k\|(o, o)}{2k \Delta^{2d+k}}.$$

We may take a large n limit supremum of the left hand side of the above for every even value of k . In the limit as $n \rightarrow \infty$, the left hand side is at most $\log \rho(\mathbf{G}) - \log \rho(T)$. This is because $\mathbf{E} \log |W| \leq \log \mathbf{E} |W|$ by concavity of \log and, as argued in Lemma 2.1, $\mathbf{E} \log |N|_{nk}(\mathbf{G}, o)$ is the average over a finitely supported probability measure (on at most m points) and each term in this average converges to $\log \rho(T)$ after division by nk and letting n tend to infinity. The inequality from the lemma now follows due to k being an arbitrary even integer. \blacksquare

3.4. Completion of the proof. Let (H, v) be a rooted and connected graph. Given integers k and ℓ , let us say H contains a bouquet if it has two disjoint ℓ -cycles C_1 and C_2 such that if the distance from v to C_j is r_j , then $k \geq \ell + \max\{r_1, r_2\}$. The situation is pictured below.

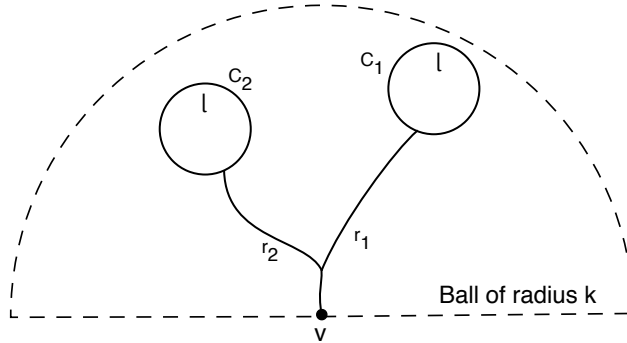


FIGURE 2. A bouquet around the root.

Suppose (H, v) contains a bouquet for the parameter values k and ℓ . They provide two closed walks in $W_{2k}(v, v)$, say P_1 and P_2 , in the following way. The walk P_j is obtained by walking from v to the closest vertex on C_j , traversing the cycle, then walking back to v along the reverse of the initial segment and appending some purely backtracking walk at the end to ensure $2k$ steps in total.

Recall the walks in $W_{2k}(v, v)$ map to a set $\overline{W_{2k}(v, v)} \subset \pi(H, v)$ by homotopy equivalence. The walks P_1 and P_2 then correspond to two mutually free elements in $\pi(H, v)$. Let Γ be the subgroup of $\pi(H, v)$ generated by $\overline{W_{2k}(v, v)}$. It is a finitely generated free group of rank at least 2. Recall that the uniform measure on $W_{2k}(v, v)$ pushes forward to a measure on $\overline{W_{2k}(v, v)}$, which induces a symmetric random walk on Γ whose Markov operator is

denoted M . The step distribution of the walk assigns positive probability to every element of $\overline{W_{2k}(v, v)}$. So by Kesten's Theorem, specifically [13, Corollary 3], the operator norm

$$(3.4) \quad \|M\| < 1.$$

As a result of (3.4), the following lemma implies

$$\sup_{k \geq 1} \frac{\mathbf{E} - \log \|M_{2k}\|(o, o)}{4k \Delta^{2d+2k}} > 0,$$

from which the proof of Theorem 3 follows by Lemma 3.3.

Lemma 3.4. *Let (\mathbf{G}, o) be an infinite sofic graph such that for some ℓ ,*

$$\Pr[o \text{ lies in an } \ell - \text{cycle of } \mathbf{G}] > 0.$$

Then there is a deterministic integer k such that, with positive probability, (\mathbf{G}, o) contains a bouquet with respect to the parameters k and ℓ .

Proof. Let $N_R(v)$ be the number of ℓ -cycles in a graph H such that at least one of its vertices is within distance R of vertex v . We will show below that for (\mathbf{G}, o) ,

$$\mathbf{E} N_R(o) \geq (R/\ell) \Pr[o \text{ lies in an } \ell - \text{cycle of } \mathbf{G}].$$

Assuming this, we may choose an R in terms of ℓ such that $\mathbf{E} N_R(o) \geq \ell \Delta^\ell + 2$. In this case, with positive probability, there are at least $\ell \Delta^\ell + 2$ different ℓ -cycles in \mathbf{G} within distance R of the root o . Whenever this happens there must be two disjoint ℓ -cycles within distance R of the root, as we explain below. We may take k to be $R + \ell$.

The reason there are two disjoint ℓ -cycles is that if H has maximal degree Δ , and v is any vertex, then there can be at most Δ^ℓ different ℓ -cycles that pass through v . This means any specific ℓ -cycle can meet at most $\ell \Delta^\ell$ other ℓ -cycles. So when there are $\ell \Delta^\ell + 2$ different ℓ -cycles, some two among them are disjoint.

In order to get the lower estimate on $\mathbf{E} N_R(o)$ consider the function

$$F(H, u, v) = \mathbf{1} \{ \text{dist}(u, v) \leq R \text{ and } u \text{ lies in an } \ell - \text{cycle of } H \}.$$

Then,

$$\sum_{u \in H} F(H, u, v) = \# \{ \text{vertices in } \ell - \text{cycles of } H \text{ within distance } R \text{ of } v \} \leq \ell N_R(v),$$

and

$$\sum_{v \in H} F(H, u, v) = |B_R(H, u)| \mathbf{1} \{ u \text{ lies in an } \ell - \text{cycle of } H \}.$$

Since (\mathbf{G}, o) is infinite almost surely, $|B_R(\mathbf{G}, o)| \geq R$. The mass transport principle then provides the lower bound on $\mathbf{E} N_R(o)$ as displayed above. \blacksquare

4. A SPECTRAL GAP THEOREM FOR FINITE GRAPHS

In this section we prove Theorem 2. Let G be a finite and connected graph with universal cover T and cover map π . Since $\lambda_1(G)$ is also the largest eigenvalue of G in absolute value, we denote it $\rho(G)$ henceforth.

For a graph H and $x \in \ell^2(H)$, let

$$(4.1) \quad f_H(x) = 2 \sum_{\{u,v\} \in H} x_u x_v,$$

where the summation is over the edges of H counted with multiplicity as there may be multi-edges and loops (recall a loop contributes degree 2 to its vertex). Thus,

$$\rho(T) = \sup_{\substack{x \in \ell^2(T) \\ \|x\|=1}} |f_T(x)| \quad \text{and} \quad \rho(G) = \sup_{\substack{x \in \ell^2(G) \\ \|x\|=1}} |f_G(x)|.$$

Theorem 2 follows from the Propositions 4.1 and 4.2 given below.

4.1. Spectral radius of an unicyclic graph.

Proposition 4.1. *Let G be a finite and connected graph with at most one cycle. Then $\rho(G) = \rho(T)$.*

Proof. There is nothing to prove if G is a tree, so assume that G has exactly one cycle (possibly a loop, or a 2-cycle made by a pair of multi-edges). We give an explicit description of T in terms of G .

Let the unique cycle in G consist of vertices v_1, \dots, v_n , in that order. Let H be the graph obtained by deleting edge (v_n, v_1) from G . We construct countably infinite copies $\dots, H_{-1}, H_0, H_1, \dots$ of H , indexed by \mathbb{Z} . For each $k \in \mathbb{Z}$, we draw an edge between v_n in H_k and v_1 in H_{k+1} . The resulting graph is T .

Let $y \in \ell^2(G)$ be the maximal eigenvector of G , normalized to $\|y\| = 1$ and with positive entries. Thus, $f_G(y) = \rho(G)$. We will construct an $x \in \ell^2(T)$, with $\|x\| = 1$, such that $f_T(x)$ approximates $\rho(G)$ arbitrarily closely.

Fix an arbitrary $N \in \mathbb{N}$. For $v' \in H_1, \dots, H_N$, set $x_{v'} = \frac{1}{\sqrt{N}} y_{\pi(v')}$. For all other $v' \in T$, set $x_{v'} = 0$.

It is evident that $\|x\| = \|y\| = 1$. Moreover,

$$f_T(x) = 2 \sum_{\substack{(u',v') \in T \\ u',v' \in H_1 \cup \dots \cup H_N}} \frac{1}{N} y_{\pi(u')} y_{\pi(v')}.$$

For each edge $(u, v) \in G$ the term $\frac{1}{N}y_u y_v$ appears N times in the above sum, except for $\frac{1}{N}y_{v_1} y_{v_n}$, which appears $N - 1$ times. Therefore,

$$f_T(x) = 2 \sum_{(u,v) \in G} y_u y_v - \frac{2}{N} y_{v_1} y_{v_n} = \rho(G) - \frac{2}{N} y_{v_1} y_{v_n}.$$

As N was arbitrary, the error term $\frac{2}{N} y_{v_1} y_{v_n}$ can be made arbitrarily small. \blacksquare

4.2. Spectral gap for a multi-cyclic graph.

Proposition 4.2. *Let G be a finite and connected graph with at least two cycles. Then $\rho(T) < \rho(G)$.*

For the remainder of this section we assume G is a finite and connected graph with at least two cycles (which may intersect, may be loops, or cycles made by multi-edges).

The *2-core* of G is defined by the following procedure. If G has at least one leaf, pick an arbitrary leaf and delete it. This operation may produce more leaves. Repeat the leaf removal operation until there are no leaves. The resulting subgraph of G is its 2-core.

The 2-core of a graph is non-empty if and only if it contains a cycle. Moreover, all cycles are preserved in its 2-core. Consequently, since G has two distinct cycles, so does its 2-core.

Let G_{int} denote the 2-core of G . Let V_{int}^G denote the vertices of G_{int} . Let E_{int}^G be the edges of G_{int} *directed* both ways, so that every edge $\{u, v\} \in G_{\text{int}}$ becomes two directed edges (u, v) and (v, u) in E_{int}^G .

Denote by V_{ext}^G the vertices of $G \setminus G_{\text{int}}$. Let E_{ext}^G be the edges of $G \setminus G_{\text{int}}$ such that they are *directed away* from the 2-core. This is possible because for every edge $\{u, v\}$ in $G \setminus G_{\text{int}}$, there is a unique shortest path from G_{int} that terminates at $\{u, v\}$. The orientation of $\{u, v\}$ is then in the direction this path enters the edge. The figure below gives an illustration of these definitions.

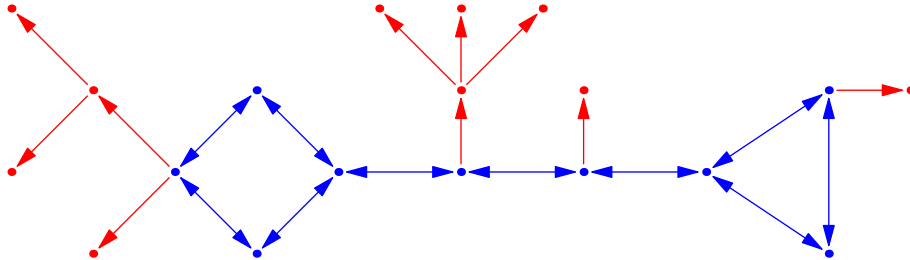


FIGURE 3. An example illustrating the definitions of V_{int}^G , V_{ext}^G , E_{int}^G , and E_{ext}^G . V_{int}^G and E_{int}^G are coloured blue. V_{ext}^G and E_{ext}^G are coloured red.

Lemma 4.1. *There exists a function $\Gamma : E_{\text{int}}^G \rightarrow [1, 2)$ such that for each directed edge $(u, v) \in E_{\text{int}}^G$,*

$$\sum_{\substack{w: (v,w) \in E_{\text{int}}^G \\ w \neq u}} \Gamma(v, w) > \Gamma(u, v).$$

Note Γ is not symmetric, i.e. $\Gamma(u, v)$ need not equal $\Gamma(v, u)$.

Proof. Every vertex of G_{int} has degree at least 2 within this subgraph. The following property of G_{int} is crucial: since G_{int} has at least two cycles and is connected, every cycle of G_{int} contains a vertex of degree more than 2.

For each directed edge $(u, v) \in E_{\text{int}}^G$ with $\deg u > 2$, set $\Gamma(u, v) = 1$. The remaining values of $\Gamma(u, v)$ will correspond to directed edges (u, v) with $\deg u = 2$. We assign these values by the following iterative procedure.

Fix an $\epsilon > 0$ to be determined. If $\deg u = 2$, u is adjacent to v_1 and v_2 , and $\Gamma(v_1, u)$ has been assigned, assign $\Gamma(u, v_2) = \Gamma(v_1, u) + \epsilon$. Due to the aforementioned crucial property, this procedure assigns a value of the form $1 + m\epsilon$ to every $\Gamma(u, v)$ with $(u, v) \in E_{\text{int}}^G$. Finally, since G is finite we may choose ϵ small enough such that Γ is strictly less than 2 everywhere.

Now if $(u, v) \in E_{\text{int}}^G$ and $\deg v > 2$,

$$\sum_{\substack{w: (v,w) \in E_{\text{int}}^G \\ w \neq u}} \Gamma(v, w) \geq 2 > \Gamma(u, v).$$

If $(u, v) \in E_{\text{int}}^G$ and $\deg v = 2$,

$$\sum_{\substack{w: (v,w) \in E_{\text{int}}^G \\ w \neq u}} \Gamma(v, w) = \Gamma(u, v) + \epsilon > \Gamma(u, v).$$

■

Lemma 4.2. *There exists a function $\Delta : E_{\text{ext}}^G \rightarrow (0, 1]$ such that for each directed edge $(u, v) \in E_{\text{ext}}^G$,*

$$\sum_{w: (v,w) \in E_{\text{ext}}^G} \Delta(v, w) < \Delta(u, v).$$

Proof. The edges in E_{ext}^G form trees, rooted at vertices in V_{int}^G and directed toward the leaves.

For each edge $(u, v) \in E_{\text{ext}}^G$, where $u \in V_{\text{int}}^G$, set $\Delta(u, v) = 1$. Assign the remaining variables by recursing down the trees in the following way. If $\Delta(u, v)$ has been assigned and v has d out-edges $(v, w) \in E_{\text{ext}}^G$, set $\Delta(v, w) = \frac{1}{d+1} \Delta(u, v)$ for each out-edge (v, w) . Then,

$$\sum_{w: (v,w) \in E_{\text{ext}}^G} \Delta(v, w) = \frac{d}{d+1} \Delta(u, v) < \Delta(u, v),$$

so the desired inequality holds. ■

Proof of Proposition 4.2. Let y be the eigenvector of the maximal eigenvalue of G chosen such that all its entries are positive and $\|y\| = 1$. Note this identity for every vertex $u \in G$:

$$(4.2) \quad \sum_{v: \{u,v\} \in G} \frac{y_v}{y_u} = \rho(G).$$

Root T at any vertex r such that $\pi(r) \in V_{\text{int}}^G$. For the rest of this proof, when we refer to an edge $(u, v) \in T$ the first vertex u is the parent, that is, closer to the root than v .

Let V_{int}^T be the vertices in T with infinitely many descendants, and V_{ext}^T be the vertices in T with finitely many descendants. Let E_{int}^T denote the edges $(u, v) \in T$ with $v \in V_{\text{int}}^T$, and E_{ext}^T the edges $(u, v) \in T$ with $v \in V_{\text{ext}}^T$.

Observe that $u \in V_{\text{int}}^T$ (resp. V_{ext}^T) if and only if $\pi(u) \in V_{\text{int}}^G$ (resp. V_{ext}^G). Similarly, $(u, v) \in E_{\text{int}}^T$ if and only if $(\pi(u), \pi(v)) \in E_{\text{int}}^G$, and $(u, v) \in E_{\text{ext}}^T$ if and only if $(\pi(u), \pi(v)) \in E_{\text{ext}}^G$. In the latter case it is crucial that u is the parent of v ; this requires v to be farther than u from V_{int}^T , so $\pi(v)$ is farther than $\pi(u)$ from V_{int}^G . Thus $(\pi(u), \pi(v))$ has the necessary orientation of an edge in E_{ext}^G .

Consider the functions Γ and Δ from Lemmas 4.1 and 4.2. Let $\gamma, \delta > 0$ be (small) constants to be determined later. Throughout the following argument we will use that

$$2|ab| \leq \eta a^2 + \eta^{-1} b^2 \quad \text{for } \eta > 0.$$

For each edge $(u, v) \in E_{\text{int}}^T$, we have

$$(4.3) \quad 2|x_u x_v| \leq \frac{y_{\pi(v)}}{y_{\pi(u)}} \left(1 + \frac{\Gamma(\pi(u), \pi(v))\gamma}{y_{\pi(u)} y_{\pi(v)}} \right)^{-1} x_u^2 + \frac{y_{\pi(u)}}{y_{\pi(v)}} \left(1 + \frac{\Gamma(\pi(u), \pi(v))\gamma}{y_{\pi(u)} y_{\pi(v)}} \right) x_v^2.$$

Analogously, for each edge $(u, v) \in E_{\text{ext}}^T$,

$$(4.4) \quad 2|x_u x_v| \leq \frac{y_{\pi(v)}}{y_{\pi(u)}} \left(1 + \frac{\Delta(\pi(u), \pi(v))\delta}{y_{\pi(u)} y_{\pi(v)}} \right) x_u^2 + \frac{y_{\pi(u)}}{y_{\pi(v)}} \left(1 + \frac{\Delta(\pi(u), \pi(v))\delta}{y_{\pi(u)} y_{\pi(v)}} \right)^{-1} x_v^2.$$

The quantity $\Delta(\pi(u), \pi(v))$ is defined because $(\pi(u), \pi(v))$ has the correct orientation of an edge in E_{ext}^G , as noted above.

Recall f_T from (4.1). The estimates (4.3) and (4.4) imply that

$$(4.5) \quad |f_T(x)| \leq 2 \sum_{\{u,v\} \in T} |x_u x_v| \leq \sum_{u \in T} g(u) x_u^2,$$

where $g(u)$ is as follows. Let $\text{pa}(u)$ denote the parent of vertex $u \in T$ and $\text{ch}(u)$ denote the set of all children of u . If $u \in V_{\text{int}}^T$ then

$$g(u) = \frac{y_{\pi(\text{pa}(u))}}{y_{\pi(u)}} \left(1 + \frac{\Gamma(\pi(\text{pa}(u)), \pi(u))\gamma}{y_{\pi(\text{pa}(u))}y_{\pi(u)}} \right) + \sum_{c \in \text{ch}(u) \cap V_{\text{int}}^T} \frac{y_{\pi(c)}}{y_{\pi(u)}} \left(1 + \frac{\Gamma(\pi(u), \pi(c))\gamma}{y_{\pi(u)}y_{\pi(c)}} \right)^{-1} \\ + \sum_{d \in \text{ch}(u) \cap V_{\text{ext}}^T} \frac{y_{\pi(d)}}{y_{\pi(u)}} \left(1 + \frac{\Delta(\pi(u), \pi(d))\delta}{y_{\pi(u)}y_{\pi(d)}} \right).$$

If $u \in V_{\text{ext}}^T$ then

$$g(u) = \frac{y_{\pi(\text{pa}(u))}}{y_{\pi(u)}} \left(1 + \frac{\Delta(\pi(\text{pa}(u)), \pi(u))\delta}{y_{\pi(\text{pa}(u))}y_{\pi(u)}} \right)^{-1} + \sum_{d \in \text{ch}(u)} \frac{y_{\pi(d)}}{y_{\pi(u)}} \left(1 + \frac{\Delta(\pi(u), \pi(d))\delta}{y_{\pi(u)}y_{\pi(d)}} \right).$$

Due to (4.5), the proposition will be proved by showing that $g(u)$ is uniformly bounded away from $\rho(G)$ over all vertices u . We separately consider the two cases $u \in V_{\text{int}}^T$ and $u \in V_{\text{ext}}^T$.

Suppose $u \in V_{\text{int}}^T$. Then for all sufficiently small $\gamma > 0$ we have the bound

$$(4.6) \quad \sum_{c \in \text{ch}(u) \cap V_{\text{int}}^T} \frac{y_{\pi(c)}}{y_{\pi(u)}} \left(1 + \frac{\Gamma(\pi(u), \pi(c))\gamma}{y_{\pi(u)}y_{\pi(c)}} \right)^{-1} \leq \sum_{c \in \text{ch}(u) \cap V_{\text{int}}^T} \frac{y_{\pi(c)}}{y_{\pi(u)}} \left(1 - \frac{\Gamma(\pi(u), \pi(c))\gamma}{y_{\pi(u)}y_{\pi(c)}} \right) + C_u \gamma^2,$$

for some constant $C_u \geq 0$ depending on u .

The terms in (4.6) depend only on the vertices $\pi(u)$ and $\pi(c)$ for $c \in \text{ch}(u)$. These are vertices of G and, since G is finite, there are only finitely many distinct values of C_u . Let C be the maximum of the C_u s. In the inequality (4.6) we may replace every C_u by C , as we do henceforth.

Inequality (4.6) implies the following bound for every $u \in V_{\text{int}}^T$ and all sufficiently small $\gamma > 0$.

$$(4.7) \quad g(u) \leq \rho(G) + \frac{\gamma}{y_{\pi(u)}^2} \left[\Gamma(\pi(\text{pa}(u)), \pi(u)) - \sum_{c \in \text{ch}(u) \cap V_{\text{int}}^T} \Gamma(\pi(u), \pi(c)) \right] \\ + \frac{\delta}{y_{\pi(u)}^2} \sum_{d \in \text{ch}(u) \cap V_{\text{ext}}^T} \Delta(\pi(u), \pi(d)) + C\gamma^2.$$

This is obtained by substituting (4.6) into the definition of $g(u)$, then multiplying out the terms and simplifying the sums by using the eigenvector equation (4.2).

By Lemma 4.1,

$$\Gamma(\pi(\text{pa}(u)), \pi(u)) - \sum_{c \in \text{ch}(u) \cap V_{\text{int}}^T} \Gamma(\pi(u), \pi(c)) < 0$$

for every $u \in V_{\text{int}}^T$. Moreover, as u ranges over V_{int}^T the quantities

$$u \mapsto \frac{1}{y_{\pi(u)}^2} \left[\Gamma(\pi(\text{pa}(u)), \pi(u)) - \sum_{c \in \text{ch}(u) \cap V_{\text{int}}^T} \Gamma(\pi(u), \pi(c)) \right]$$

are determined by the graph G . So they attain finitely many values and have a maximum value $C_{\text{int}} < 0$. Analogously, the quantities

$$(4.8) \quad u \mapsto \frac{1}{y_{\pi(u)}^2} \sum_{d \in \text{ch}(u) \cap V_{\text{ext}}^T} \Delta(\pi(u), \pi(d))$$

have a maximum value $D_{\text{int}} \geq 0$ as u ranges over V_{int}^T . So we infer that for every $u \in V_{\text{int}}^T$ and all sufficiently small $\gamma > 0$,

$$(4.9) \quad g(u) \leq \rho(G) + C_{\text{int}}\gamma + C\gamma^2 + D_{\text{int}}\delta.$$

Now suppose that $u \in V_{\text{ext}}^T$. By an analogous argument as above, there exists a constant $D \geq 0$ independently of u such that for all sufficiently small $\delta > 0$,

$$g(u) \leq \rho(G) + \frac{\delta}{y_{\pi(u)}^2} \left[-\Delta(\pi(\text{pa}(u)), \pi(u)) + \sum_{d \in \text{ch}(u)} \Delta(\pi(u), \pi(d)) \right] + D\delta^2.$$

By Lemma 4.2,

$$-\Delta(\pi(\text{pa}(u)), \pi(u)) + \sum_{d \in \text{ch}(u)} \Delta(\pi(u), \pi(d)) < 0$$

for all $u \in V_{\text{ext}}^T$. Therefore, as before, there is a $D_{\text{ext}} < 0$ such that for every $u \in V_{\text{ext}}^T$ and all sufficiently small $\delta > 0$,

$$(4.10) \quad g(u) \leq \rho(G) + D_{\text{ext}}\delta + D\delta^2.$$

Finally, we select $\gamma > 0$ small enough that (4.9) holds and $C_{\text{int}}\gamma + C\gamma^2 < 0$. This is possible because $C_{\text{int}} < 0$. Then we select $\delta > 0$ small enough such that (4.10) holds while both $C_{\text{int}}\gamma + C\gamma^2 + D_{\text{int}}\delta < 0$ and $D_{\text{ext}}\delta + D\delta^2 < 0$. This is possible due to the choice of γ and because $D_{\text{ext}} < 0$.

In light of (4.9) and (4.10), our choice of γ and δ above imply that there is an $\epsilon > 0$ such that for every vertex $u \in T$, $g(u) \leq \rho(G) - \epsilon$. This implies $\rho(T) < \rho(G)$. \blacksquare

5. FUTURE DIRECTIONS

It would be interesting to find an effective version of Theorem 1 in the following sense. Let G_1, G_2, G_3, \dots be finite, connected graphs with $|G_n| \rightarrow \infty$. Suppose they have a common universal cover T , and are Ramanujan in that all but their largest eigenvalue are at most $\rho(T)$ in absolute value. What is the “essential girth” of G_n in terms of its size, which is to say, the asymptotic girth after possibly having removed at order of $o(|G_n|)$ edges? For

d -regular Ramanujan graphs it is known, see [1], that the essential girth is at least of order $\log \log |G|$. All known constructions provide graphs with girth of order $\log |G|$. It seems that a lower bound of order $\log |G|$ for the girth is unknown even when G is a Cayley graph.

It would also be interesting to find an effective form of Theorem 2 in terms of the size and the maximal degree of G . The theorem is in some ways an analogue of Theorem 3 for finite graphs, although, its word-for-word reformulation is false for infinite graphs. In this regard it would be interesting to prove a spectral gap between $\rho(G)$ and $\rho(T)$ under natural hypotheses on an infinite graph G . For instance, to prove an effective spectral gap when there is an R such that the R -neighbourhood of every vertex in G contains a cycle.

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